HYPER-MATHEMATICS

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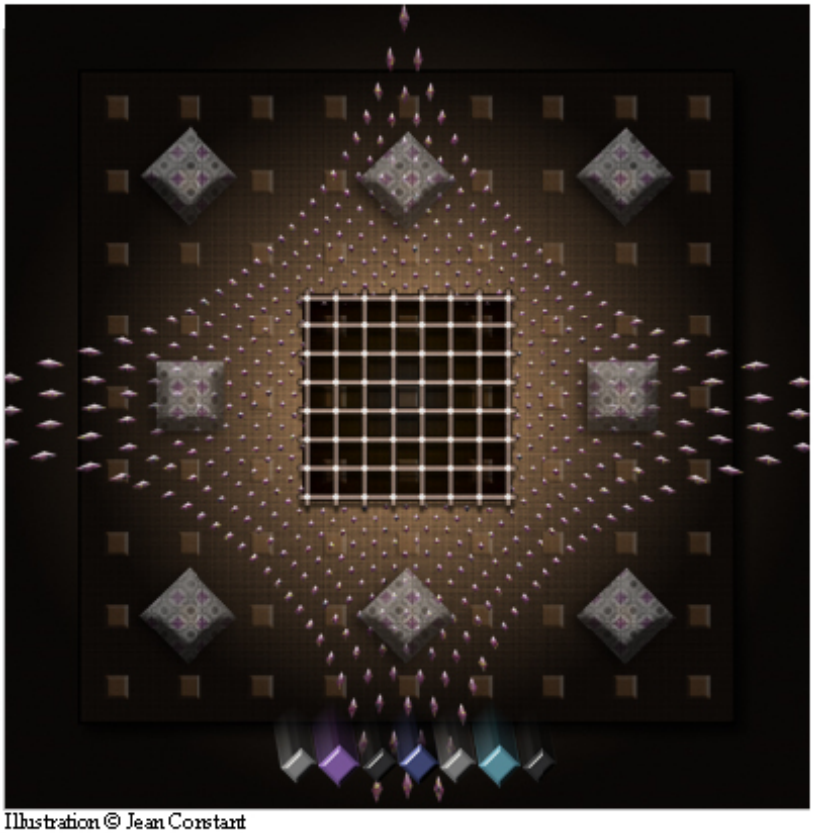
SET THEORY



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SET THEORY

Set theory is the branch of mathematics that studies sets, which are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics.

The modern study of set theory was initiated by Cantor and Dedekind in the 1870s. After the discovery of

The momentum of set theory was such that debate on the paradoxes did not lead to its abandonment. The work of Zermelo in 1908 and Fraenkel in 1922 resulted in the canonical axiomatic set theory ZFC, which is thought to be free of paradoxes. The work of analysts such as Lebesgue demonstrated the great mathematical utility of set theory. Axiomatic set theory has become woven into the very fabric of mathematics as we know it today.

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Axiomatic set theory

Elementary set theory can be studied informally and intuitively, and so can be taught in primary schools using, say, Venn diagrams. The intuitive approach silently assumes that all objects in the universe of discourse satisfying any defining condition form a set. This assumption gives rise to antinomies, the simplest and best known of which being Russell's paradox. Axiomatic set theory was originally devised to rid set theory of such antinomies.

The most widely studied systems of axiomatic set theory imply that all sets form a cumulative hierarchy. Such systems come in two flavors, those whose ontology consists of:

- *Sets alone*. This includes the most common axiomatic set theory, Zermelo–Fraenkel set theory (ZFC), which includes the axiom of choice. Fragments of ZFC include:
  - Zermelo set theory, which replaces the axiom schema of replacement with that of separation;
  - General set theory, a small fragment of Zermelo set theory sufficient for the Peano axioms and finite sets;
  - Kripke-Platek set theory, which omits the axioms of infinity, powerset, and choice, and weakens the axiom schemata of separation and replacement.
- *Sets and proper classes*. This includes Von Neumann-Bernays-Gödel set theory, which has the same strength as ZFC for theorems about sets alone, and Morse-Kelley set theory, which is stronger than ZFC.

For the above systems, allowing urelements (objects that can be members of sets while having no members themselves) does not give rise to any interesting mathematics.

paradoxes in informal set theory, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms, with the axiom of choice, are the best-known.

Set theory, formalized using first-order logic, is the most common foundational system for mathematics. The language of set theory is used in the definitions of nearly all mathematical objects, such as functions, and concepts of set theory are integrated throughout the mathematics curriculum. Elementary facts about sets and set membership can be introduced in primary school, along with Venn diagrams, to study collections of commonplace physical objects. Elementary operations such as set union and intersection can be studied in this context. More advanced concepts such as cardinality are a standard part of the undergraduate mathematics curriculum.

Beyond its use as a foundational system, set theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

History

See Johnson (1972) for a book-length treatment. Mathematical topics typically emerge and evolve through interactions among many researchers. The point of origin of set theory is somewhat unusual in that it can be identified as an 1874 paper by Georg Cantor: "On a Characteristic Property of All Real Algebraic Numbers".[1][2][3]

Beginning with the work of Zeno around 450 BC, mathematicians had been struggling with the concept of infinity. Especially notable is the work of Bernard Bolzano in the first half of the 19th century. The modern understanding of infinity began 1867-71, with Georg Cantor's work on number theory. An 1872 meeting between Cantor and Dedekind much influenced Cantor's thinking and culminated in Cantor (1874)[1].

Cantor's work initially polarized the mathematicians of his day. While Weierstrass and Dedekind supported Cantor, Kronecker, now seen as a founder of mathematical constructivism, did not. But the utility of Cantorian concepts such as one-to-one correspondence among sets, his proof that there are more real numbers than integers, and the "infinity of infinities" ("Cantor's paradise") the power set operation gives rise to, eventually led to the widespread acceptance of Cantorian set theory.

The next wave of excitement in set theory came around 1900, when it was discovered that Cantorian set theory gave rise to several contradictions, called antinomies or paradoxes. Russell and Zermelo independently found the simplest and best known paradox, now called Russell's paradox and involving "the set of all sets that are not members of themselves." Clearly this set cannot be a member of itself, and hence it must be a member of itself! In 1899 Cantor had himself posed the question: "what is the cardinal number of the set of all sets?" and obtained a related paradox. It was later realized that these paradoxes are not merely set theoretic, and that in logic the sentence "this sentence is false" gives rise to a similar problem, for if the sentence is true, it must be false. Kurt Gödel used this fact in the 1931 proof of his celebrated incompleteness theorem.